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Exact relations between critical exponents for elastic stiffness and electrical conductivity of two-dimensional percolating networks

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It has long been known that the critical exponent T of the elastic stiffness $C_e \propto \Delta p^T$ of a d-dimensional percolating network ($\Delta p \equiv p - p_c > 0$ measures the closeness of the network to its percolation threshold p_c) satisfies the following inequalities: $1 + d\nu \leqslant T \leqslant t + 2\nu$, where t is the critical exponent of the electrical conductivity $\sigma_e \propto \Delta p^t$ of the same network and ν is the critical exponent of the percolation correlation length $\xi \propto \Delta p^{-\nu}$. Similarly, the critical exponents that characterize the divergences $C_e \propto |\Delta p|^{-S}$, $\sigma_e \propto |\Delta p|^{-s}$ of a rigid or normal and a superconducting or normal random mixture ($\Delta p \equiv p - p_c < 0$ now measures the closeness of the rigid or superconducting constituent to its percolation threshold p_c) have long been known to satisfy $S \leqslant s$. We now show that, when $d = 2$, T is in fact exactly equal to $t + 2\nu$ and S is exactly equal to s.

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I. INTRODUCTION

That the macroscopic stiffness moduli of a percolating network of elastic bonds behave differently from the electrical conductivity of a similar network of conducting bonds was first pointed out by the authors of Refs. [1,2]. By numerical simulations of two-dimensional (2D) random elastic networks it was shown, shortly thereafter, that the critical exponents for those two material coefficients differ considerably [3,4],

$$\sigma_e \propto \Delta p^t, \quad C_e \propto \Delta p^T, \quad t \cong 1.30, \quad T \cong 4.0,$$

where σ_e is the macroscopic electrical conductivity, C_e is a generic macroscopic elastic stiffness coefficient, $\Delta p \equiv p - p_c > 0$ is the difference between the volume fraction or bond occupation probability p and its value at the percolation threshold p_c, t and T are the critical exponents.

These results apply to the case where the basic elastic properties of the network bonds include a ‘‘bond stretching force’’ $k\delta b$ that is needed to change the bond length by δb, as well as an ‘‘angle bending force’’ $m\delta\varphi$ that is needed in order to change the angle between two neighboring bonds (i.e., bonds connected to the same network site) by $\delta\varphi$. Only if both types of basic elastic responses are present, with nonzero force constants k and m, does the rigidity threshold of the system (i.e., the value of p at which the macroscopic bulk and shear moduli all tend to 0) coincide with the usual percolation threshold p_c, which is also the electrical conductivity threshold where $\sigma_e \to 0$. Elastic networks where only bond stretching forces appear, (so-called ‘‘central force models’’) i.e., every bond represents a simple one-dimensional spring, do not only have a rigidity threshold that differs from p_c, [in the case of a 2D simple square or a 3D (three-dimensional) simple cubic network of springs, even the undiluted network is nonrigid] but their critical behavior near that threshold is apparently also different, with different values for the critical exponents ν and T [5,6].

Percolating continuum elastic composites apparently lie in the universality class of the networks with both types of forces. This is evident from physical considerations, namely, when voids are created in an elastic solid, even a tenuous chain of narrow solid sections locally resists bending at any point, as well as resisting stretching. This is also consistent with the fact that such a system will fall apart mechanically, with all the stiffness moduli going to zero, simultaneously with the conductivity decreasing to zero, i.e., the electrical conductivity and elastic rigidity thresholds coincide in any real continuum composite. Finally, experiments on randomly perforated thin films have shown that the critical exponent T is close to predictions resulting from the 2D network models with both types of forces, and is inconsistent with the predictions resulting from central force models [7–9]. Here, we only consider networks where bond stretching forces and angle bending forces are both present.

In the case of a random network of elastic bonds and totally rigid bonds, i.e., bonds for which both types of force constants k, m are infinite, (such mixtures are often characterized by using the abbreviated adjective ‘‘superelastic’’) the macroscopic elastic stiffness moduli diverge when the fraction of rigid bonds approaches its percolation threshold p_c *from below*. This can be compared with the behavior of a similar network of normal conducting bonds and perfectly conducting or superconducting bonds, (such mixtures are often termed ‘‘superconducting’’ for brevity) where the macroscopic conductivity diverges when the fraction of the latter bonds approaches p_c from below. Denoting the fraction of rigid or superconducting bonds by p, and using $\Delta p \equiv p - p_c < 0$, the critical behaviors of the macroscopic moduli in 2D random networks are

$$\sigma_e \propto |\Delta p|^{-s}, \quad C_e \propto |\Delta p|^{-S}, \quad s = t \cong 1.30.$$

The fact that $s = t$ exactly is a consequence of classical duality [10,11].

Interestingly, the best numerical value obtained for T in simulations of 2D random networks, namely, $T = 3.96 \pm 0.04$ [4], is in excellent agreement with the best numerical value for the upper bound, namely, $t + 2\nu = 3.964 \pm 0.007$. This naturally raises the challenge to try and see whether a rigorous equality $T = t + 2\nu$ can be proven to hold. Such a proof is presented in this paper. While the conjecture that T

$=t+2\nu$ exactly has been expressed before [12,13], a convincing proof has never been formulated, as far as I know.

Results for the critical exponent S were also obtained from numerical simulations. When a 2D random network model with bond stretching and angle bending forces was used, the results were $S \cong s = 1.30 \pm 0.01$ [14]. This approximate equality $S \cong s$ of the superelastic and superconducting exponents again raises the challenge to try and see whether a rigorous equality $S=s$ can be proven. Such a proof is also presented in this article.

Simulations of another superelastic network model, where flat circular disks are placed on a square lattice and connected by springs in such a way that the 2D rigidity threshold coincides with the percolation threshold, yielded a different result, namely, $S=1.02 \pm 0.07$ [15], which is less than the previous result by about 4 standard deviations. Another set of simulations, using a network model much like the one used in Ref. [14], yielded $S=1.23 \pm 0.03$ [16]. Although this is closer to the best value for $s=1.297 \pm 0.007$ [17–19], it is still lower than that value by about 2 standard deviations. These discrepancies have not been resolved until now.

A detailed discussion of elastic network models, mostly based on the Koch curve fractal, has led Limat to conclude that the conjectures $T=t+2\nu$ and $S=s$ are incorrect [20]. In contrast with that conclusion, I believe that the results presented here show that those equalities should hold generically in the case of 2D networks, i.e., for a percolating network as well as for other types of fractal networks.

In parallel with the numerical studies described above, further light was shed on these problems starting with the pioneering work of Kantor and Webman [21]. They showed that, in a percolating network with both bond stretching and angle bending forces, the bond stretching becomes unimportant and irrelevant close to p_c. All the macroscopic moduli then depend only on the angle bending force constants. This is due to the fact that, as p_c is approached, the percolating cluster becomes increasingly tenuous. Consequently, any kind of macroscopic strain can be produced with increasingly greater efficiency (i.e., at a decreasing relative cost in energy) by bending of interbond angles as compared to stretching of individual bonds. This picture also leads to a rigorous lower bound for T, namely [21]

$$1+d\nu \leqslant T, \tag{1}$$

where d is the dimensionality of the network and ν is the percolation correlation length critical exponent

$$\xi \propto \Delta p^{-\nu},$$

which is equal to 4/3 in 2D systems [22,23]. It is noteworthy that the concept of "singly connected bonds (SCB)" plays an important role in derivations of Eq. (1) [21,24] (an SCB between any two connected sites of a diluted network is a bond that, when deleted, severs the connection between those sites).

Somewhat later, an upper bound for T was found, namely [25]

$$T \leqslant t+2\nu. \tag{2}$$

A rigorous derivation of this inequality for the case of 2D networks appears in Sec. II below.

The fact that the bond stretching force constant k is irrelevant was confirmed by numerical simulations, which showed that the macroscopic elastic response near the percolation threshold depends only on the angle bending force constant m [3,4]. It should, however, be remembered that, for any value of Δp, this ceases to be true if the ratio k/m is too small. In that case, the critical behavior presumably crosses over to a different universality class. We will assume that the usual universality class, where k is irrelevant, is appropriate whenever the total bond stretching energy is less than the total angle bending energy.

In the case of the superelastic and superconducting critical exponents, a discussion by Milton [26] can be extended to yield the following inequality, which is valid for arbitrary d:

$$S \leqslant s. \tag{3}$$

It is noteworthy that all of the numerical results quoted above for S satisfy this inequality. A rigorous derivation of this inequality also appears in Sec. II below.

The rest of this paper is organized as follows. Section II reviews the previously known results regarding the relation between the critical exponents T and t, as well as the relation between S and s. This includes the variational principle used to obtain Eq. (2) for $d=2$, as well as the variational argument used to obtain Eq. (3). In Sec. III another variational principle is used to obtain results similar to those equations, but with the $\leqslant$ inequality sign replaced by the opposite $\geqslant$ inequality sign. Section IV discusses the possibility of generalizing the results obtained for 2D systems to arbitrary values of d.

II. REVIEW OF PREVIOUSLY KNOWN RESULTS

The analysis presented in this paper is based on the following representation for the potential energy E_L of a 2D network of elastic bonds,

$$2E_L = \sum_i k_i(\delta b_i)^2 + \sum_{(i,j)=nn} m_{ij}(\delta\varphi_i - \delta\varphi_j)^2. \tag{4}$$

Here δb_i is the change in length of the bond i while $\delta\varphi_i$ is the change in its absolute 2D orientation, i.e., a rotation angle in the system plane. The second summation ranges over all pairs of nearest neighbor bonds, i.e., bonds that are joined at a network site. This representation was first used by Kantor and Webman, who derived a bound on the critical behavior of the macroscopic stiffness moduli of such a network when it is diluted by random removal of bonds until it approaches its percolation threshold [21]. In this way they obtained the lower bound of Eq. (1) for T, which reads as follows for the 2D case:

$$T \geqslant 1+2\nu = 11/3.$$

Somewhat later, the upper bound of Eq. (2) for T was found by Roux [25], namely,

$$T \leqslant t + 2\nu.$$

Another derivation of this bound for the 2D case was later given by the present author [24], based on the expression (4) and on the variational property of the following expression for the Joule heat production rate W_L of a network of conducting bonds:

$$W_L = \sum_{(i,j)=nn} g_{ij}(V_i - V_j)^2. \tag{5}$$

Here V_i is the potential at site i of the network and g_{ij} is the electrical conductance of the bond (i,j), which connects the nearest neighbor sites i, j.

The above expression is tantalizingly similar to the second sum of Eq. (4), leading to the idea that a correspondence might be established between the elastic network problem and the conducting network problem. However, in order to establish such a correspondence we need to implement those two problems on networks that are related but different. The conductivity problem will be considered on the *covering network* of the elastic network, and the latter will also be referred to as the *original network*: The covering network is constructed by replacing each bond of the original network by a site, and connecting two such sites i and j by a bond (i,j) if and only if the original bonds were nearest neighbors [27]. Clearly, the covering network percolates if and only if the original network percolates. Furthermore, the covering network presumably belongs to the same universality class of percolation as the original network, at least as far as topological and simple electrical conductivity properties are concerned. Therefore, the relevant critical exponents will also have the same values. The sum in Eq. (5) is now similar to the second sum in Eq. (4), since each site of the covering lattice corresponds to a bond of the original lattice and each bond of the covering lattice corresponds to an angle between nearest neighbor bonds of the original lattice. The correspondence becomes even stronger if the bond conductances g_{ij} are chosen to equal the angle bending force constants m_{ij} of the corresponding interbond angles.

In a network of conductors, if the potentials V_i are fixed at some partial set of sites, which can be considered as "surface sites," then the expression for W_L of Eq. (5) attains its unique absolute minimum value when all the V_i's assume their correct physical values. Therefore, if we assume $g_{ij} = m_{ij}$ for all the bonds of the covering network, then the solution $\delta\varphi_i$ of the elastic problem on the original network provides a trial set of values for V_i on the covering network, if the boundary values of $\delta\varphi_i$ and V_i are the same. Consequently, we can then write

$$\begin{aligned} W_L &< \sum_{(i,j)=nn} m_{ij}(\delta\varphi_i - \delta\varphi_j)^2 \\ &< \sum_i k_i(\delta b_i)^2 + \sum_{(i,j)=nn} m_{ij}(\delta\varphi_i - \delta\varphi_j)^2 \\ &= 2E_L. \end{aligned} \tag{6}$$

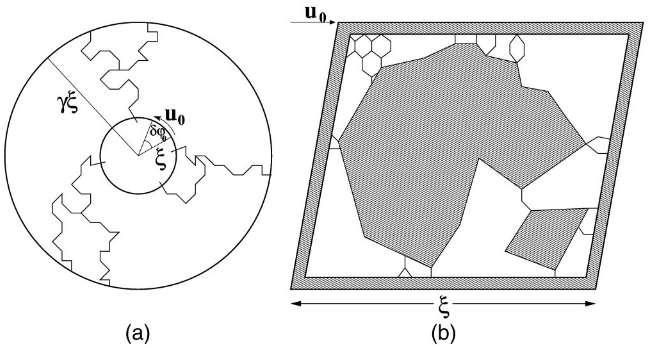

FIG. 1. (a) Flat ring configuration of a percolating network near p_c, where the inner boundary is a circle of radius ξ, which is rigidly rotated by an angle $\delta\varphi_0 = |\mathbf{u}_0|/\xi$. Thus, each of the bonds that intersect that boundary has its orientation rotated by $\delta\varphi_0$, but its length remains unchanged. The outer boundary is a concentric circle of radius $\gamma\xi$, where $\gamma \gg 1$ is a large but fixed numerical factor, i.e., it is independent of Δp, in contrast with the percolation correlation length ξ. In the covering lattice, each bond is replaced by a site, and the sites on the inner boundary have a potential equal to $\delta\varphi_0$ applied to them. Both the potential and the planar displacements at the outer boundary are made to vanish. (b) $\xi \times \xi$ square configuration of a rigid/normal network, near p_c of the rigid constituent. A displacement $\mathbf{u}_0$ is applied at the upper boundary. The gray regions inside the system represent rigid clusters. Only some of the normal (i.e., nonrigid) bonds are shown. In many cases, the separation between the largest rigid cluster and the boundary involves just one normal bond, or a small number of such bonds. Consequently, many of the boundary bonds have their lengths changed by an amount δb_i which is of order $|\delta\mathbf{u}_0|$, and their orientations changed by an angle $\delta\varphi_i$ which is of order $\delta\varphi_0 \equiv |\mathbf{u}_0|/b_0$, where b_0 is the length of an elementary unstrained bond. In the covering network, the corresponding boundary sites will have an electric potential equal to that same value $\delta\varphi_i$ applied to them.

In order to obtain an upper bound for T, we can consider an elastic network in the shape of a 2D circular ring where the inner radius is ξ and the outer radius is greater than ξ by a large but fixed factor γ that is independent of Δp—see Fig. 1(a). The bonds at the outer boundary are fixed, so that $\delta\varphi_i = \delta b_i = 0$ there, while the inner boundary is rotated rigidly in the plane by an angle $\delta\varphi_0$, so that $\delta b_i = 0$ and $\delta\varphi_i = \delta\varphi_0 \neq 0$ for bonds at that boundary. For this kind of system, ξ is the only relevant length scale, beyond the basic bond length b_0. At the length scale ξ, we can already use the macroscopic stiffness C_e to write an expression for the total elastic energy,

$$E_L \sim C_e \xi^2 (\delta\varphi_0)^2. \tag{7}$$

For the covering network of conductors $g_{ij} = m_{ij}$, with boundary condition $V_i = 0$ at the outer boundary and $V_i = \delta\varphi_0$ at the inner boundary, the total production rate of Joule heat is then of order,

$$W_L \sim \sigma_e (\delta\varphi_0)^2, \tag{8}$$

apart from a multiplicative logarithmic correction that we ignore. The factor ξ^2, which appears in the expression for E_L

but is absent from the expression for W_L, reflects the different physical dimensions of C_e (energy per unit area) and σ_e (conductance) in 2D systems. Using these expressions in the inequality (6) we get

$$C_e\xi^2>\sigma_e \Rightarrow \Delta p^{T-2\nu}>\Delta p^t \Rightarrow T\leqslant t+2\nu. \tag{9}$$

We note that if the inner radius of the ring L were much greater than ξ, then instead of Eq. (7) we would have $E_L \sim C_e L^2(\delta\varphi_0)^2$, while Eq. (8) for W_L would remain unchanged. In that case the inequality of Eq. (6) would be satisfied in a trivial fashion and become useless.

Numerical simulations of 2D percolating elastic networks found $T=3.96\pm 0.04$ [4]. This is consistent with the value 4, which is the exact value of T in a six-dimensional percolating medium [24]. It is also in excellent agreement with the value $t+2\nu=3.964\pm 0.007$, which is based upon $t=1.297 \pm 0.007$ [17–19] and on the exact result $\nu=4/3$ [22].

In the case of a continuum composite, the macroscopic moduli $\hat{\sigma}_e$, $\hat{C}_e$ have the property that, for any given volume average electric field $\mathbf{E}_0$ or strain $\hat{\epsilon}_0$, the total rate of production of Joule heat per unit volume $\mathbf{E}_0\cdot\sigma_e\cdot\mathbf{E}_0$ or twice the total elastic potential energy per unit volume $\hat{\epsilon}_0\cdot\hat{C}_e\cdot\hat{\epsilon}_0$ are the absolute minimum values of the following quadratic functionals of the local electric potential $\phi(\mathbf{r})$ and local displacement field $\mathbf{u}(\mathbf{r})$, respectively:

$$\frac{1}{V}\int_V dV\,\nabla\phi\cdot\hat{\sigma}\cdot\nabla\phi, \tag{10}$$

$$\frac{1}{V}\int_V dV\,\hat{\epsilon}\cdot\hat{C}\cdot\hat{\epsilon}, \quad \epsilon_{\alpha\beta}\equiv\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial r_\beta}+\frac{\partial u_\beta}{\partial r_\alpha}\right), \tag{11}$$

subject to the conditions that $\phi(\mathbf{r})=-(\mathbf{r}\cdot\mathbf{E}_0)$ and $\mathbf{u}(\mathbf{r}) =\hat{\epsilon}_0\cdot\mathbf{r}$ at the system boundaries. Here $\hat{\sigma}(\mathbf{r})$ is the local electrical conductivity, a second rank tensor, $\hat{C}(\mathbf{r})$ is the local elastic stiffness, a fourth rank tensor, and $\hat{\epsilon}(\mathbf{r})$ is the local strain, a second rank tensor.

We assume that the local response is isotropic, hence $\hat{\sigma}$ is a position dependent scalar and $\hat{C}$ depends on just two position dependent Lamé coefficients λ and μ

$$C_{\alpha\beta\gamma\omega}=\lambda\,\delta_{\alpha\beta}\delta_{\gamma\omega}+\mu(\delta_{\alpha\gamma}\delta_{\beta\omega}+\delta_{\alpha\omega}\delta_{\beta\gamma}). \tag{12}$$

These coefficients satisfy the following relations:

$$C_{\alpha\beta\alpha\beta}\equiv\mu>0 \quad \text{for} \quad \alpha\neq\beta,$$

$$C_{\alpha\alpha\alpha\alpha}\equiv C_{11}\equiv\lambda+2\mu=\rho v_s^2>0,$$

$$\kappa\equiv\lambda+\frac{2}{d}\mu=C_{11}-2\frac{d-1}{d}\mu>0,$$

where κ is the bulk modulus, μ is the shear modulus, C_{11} is another one of the elastic stiffness moduli, written in Voigt's notation, v_s is the longitudinal sound velocity, and ρ is the mass density of the solid. The minimizing functions $\phi(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$ are the correct local physical potential and displacement fields.

The solution $\phi(\mathbf{r})$ of an appropriate conductivity problem in the volume V can be used to construct trial functions for the elastic energy functional: Following Milton [26], we take $\mathbf{u}(\mathbf{r})=\phi_\alpha(\mathbf{r})\mathbf{e}_\beta$ as a trial function, where $\phi_\alpha(\mathbf{r})=-r_\alpha$ at the boundary and $\mathbf{e}_\beta$ denotes the unit vector along r_β. Using this trial function we easily get

$$\hat{\epsilon}\hat{C}\hat{\epsilon}=(\lambda+2\mu)\left(\frac{\partial\phi_\alpha}{\partial r_\beta}\right)^2+\mu\sum_{\gamma\neq\beta}\left(\frac{\partial\phi_\alpha}{\partial r_\gamma}\right)^2$$

$$<(\lambda+2\mu)(\nabla\phi_\alpha)^2=C_{11}(\nabla\phi_\alpha)^2,$$

and applying this inequality to the integrand of Eq. (11), we get the following results for $\hat{\epsilon}_0$ and $\hat{C}_e$:

$$\epsilon^{(0)}_{\beta\alpha}=\epsilon^{(0)}_{\alpha\beta}=-\tfrac{1}{2}(1+\delta_{\alpha\beta}), \quad \text{all the other } \epsilon^{(0)}_{\gamma\omega}=0, \tag{13}$$

$$C^{(e)}_{\alpha\beta\alpha\beta}\leqslant\frac{1}{V}\int_V dV\,C_{11}(\nabla\phi_\alpha)^2=\sigma^{(e)}_{\alpha\alpha}[\sigma(\mathbf{r})], \tag{14}$$

$$\sigma(\mathbf{r})\equiv C_{11}(\mathbf{r})\equiv\lambda(\mathbf{r})+2\mu(\mathbf{r})=\kappa(\mathbf{r})+2\frac{d-1}{d}\mu(\mathbf{r}). \tag{15}$$

This means that the macroscopic conductivities of a heterogeneous medium, where the local conductivity $\sigma(\mathbf{r})$ is equal to the local stiffness coefficient $C_{11}(\mathbf{r})$, provide *upper bounds* for some of the macroscopic stiffness moduli. In particular, if we are dealing with a composite medium that has an isotropic microstructure, where $\hat{\sigma}_e\equiv\sigma_e$ is a scalar and $\hat{C}_e$ has the form of Eq. (12), then

$$\mu_e\leqslant\sigma_e[\sigma(\mathbf{r})], \tag{16}$$

$$C^{(e)}_{11}=\lambda_e+2\mu_e\leqslant\sigma_e[\sigma(\mathbf{r})], \tag{17}$$

$$\kappa_e=C^{(e)}_{11}-2\frac{d-1}{d}\mu_e. \tag{18}$$

Applying the inequalities of Eqs. (16) and (17) in the case of a rigid/normal mixture, we immediately obtain Eq. (3). In the case of a percolating (diluted) network, these inequalities lead to $T\geqslant t$, which is a much weaker lower bound on T than the inequality of Eq. (1).

III. NEW BOUNDS FOR *T* AND *S*

We now invoke another variational principle, whereby the expression (4) is minimized only when δb_i and $\delta\varphi_i$ attain their correct equilibrium physical values, subject to fixed given values at some surface bonds and to certain constraints, which are described below. We then try to use the exact solution V_i of the electrical conductivity problem on the covering network as trial values for $\delta\varphi_i$.

In order to succeed, this approach needs to overcome two problems: (a) In order to be acceptable as trial values for

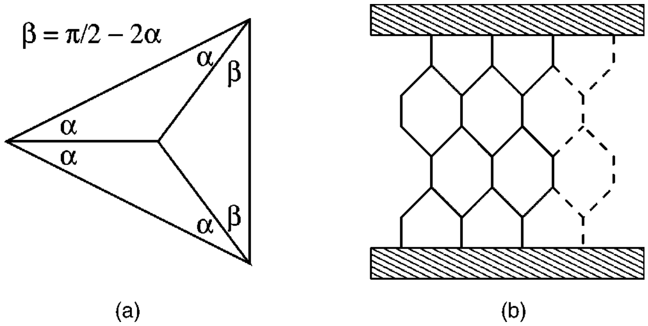

FIG. 2. (a) Compact cluster of 6 bonds where, if the orientations of the bonds are assigned arbitrarily, it is usually impossible to find any consistent bond lengths. For example, if the two small triangles on the left side are both isoceles with identical angles α, as shown, then the third small triangle must also be isoceles, which leaves no freedom in choosing the orientation of the rightmost bond. (b) Honeycomb network rigidly connected to two parallel rigid plates. When a single column of (dashed) bonds with *preassigned arbitrary orientations* is added to the network, it is always possible to find consistent bond lengths, which leave all the loops intact, if the orientations are not too different from those of the ideal honeycomb structure. Moreover, sufficient freedom remains in the choice of the δb_i's, of the bonds in each elementary hexagon, to allow the sum $\Sigma_i(\delta b_i)^2$ in each hexagon to be minimized.

$\delta\varphi_i$, the V_i's need to satisfy constraints that are absent in the electrical conductivity problem. These constraints arise because any closed loop of elastic bonds must remain a closed loop even after the bonds are distorted. (b) Even if those constraints are satisfied, we would still only have a trial value for the second sum in Eq. (4), whereas we are seeking an upper bound for the entire elastic potential energy E_L.

The first problem is overcome if values for δb_i can be found that satisfy all the constraints when $\delta\varphi_i = V_i$. The question whether it is possible to satisfy those constraints, for a given set of $\delta\varphi_i$, by an appropriate choice of δb_i depends on the types of closed loops found in the network. Clearly, a triplet of simple triangular loops, whose circumference is also a simple triangular loop [see Fig. 2(a)], does not allow those constraints to be satisfied in general. One could argue that, in the vicinity of p_c, such compact clusters of bonds are unimportant or irrelevant. Alternatively, we can limit ourselves to networks that have a more open microstructure, even when they are undiluted. For example, if we consider a honeycomb network, as in Fig. 2(b), then, even in the undiluted network, values of δb_i can always be found that satisfy all the constraints. This can be seen by considering the addition of an entire column of bonds (represented by dashed lines) to an existing honeycomb network that is rigidly connected to two parallel rigid plates, as shown in Fig. 2(b). These δb_i are all of order $b_0\delta\varphi_0 = b_0 V_0$, where b_0 is the length of an undistorted bond and $\delta\varphi_0 = V_0$ is the macroscopic potential difference that is applied across the entire sample.

Using the trial values $\delta\varphi_i = V_i$, and δb_i as determined from them by the constraints, in an elastic network where $m_{ij}\equiv g$ and $k_i\equiv k$, (the bonds with $k_i = m_{ij} = 0$ are simply not included in the explicit summations) we get the following upper bound for $2E_L$:

$$2E_L < k\sum_i (\delta b_i)^2 + g\sum_{(i,j)=nn}(V_i - V_j)^2. \tag{19}$$

We would now like to assign a value to the bond stretching constant k that will make the first sum not greater than the second sum, which is equal to W_L of the covering network. However, in general, the first sum depends on the total size of the system, as well as on Δp, while the second sum depends only on Δp, due to the dimensionality $d=2$. [This is literally true for the ring geometry of Fig. 1(a); for other 2D sample geometries W_L can depend on details of the macroscopic shape.] In order to determine a value of k that will satisfy the above requirement for all the samples that we need to consider, we assume that the system is again ring shaped, as in Fig. 1(a), with inner radius equal to ξ and a much larger outer radius $\gamma\xi$. In that case, both sums depend only on Δp. The second sum is of order $g\Delta p^t(\delta\varphi_0)^2$, i.e., it *decreases* when Δp decreases. By contrast, the first sum is of order $N_L k b_0^2(\delta\varphi_0)^2$, where N_L is the total number of bonds in the independent closed loops of the ring. That is because the density of loops of size (i.e., gyration radius) much less than ξ does not change when Δp is decreased further, while the total area of the ring increases; also, more bonds will appear in the loops of size comparable to or greater than ξ, since ξ itself increases. N_L is thus a quantity that *increases* when Δp decreases. Thus, if we choose k by requiring that

$$k\sum_i (\delta b_i)^2 = g\sum_{(i,j)=nn}(V_i - V_j)^2 = W_L \tag{20}$$

for a given positive value Δp_0 of Δp, then the left hand side of this equation will be less than the right hand side for any larger value of Δp.

We now let $\delta\varphi_i$ and δb_i vary, subject to the constraints, in order to seek the minimum value of the elastic energy. When that minimum is attained, the sum $g\Sigma_{(i,j)=nn}(\delta\varphi_i - \delta\varphi_j)^2$ will have *increased* to a value that is greater than W_L. At the same time, the sum $k\Sigma_i(\delta b_i)^2$ will have *decreased*, by an even greater amount, to less than the value W_L that it had when δb_i were determined, by the constraints, from the assigned $\delta\varphi_i = V_i$. This is dictated by the fact that the sum of those two sums will have decreased to its minimum value, subject to the constraints, hence it must be less than $2W_L$. Therefore, the two sums will now satisfy

$$k\sum_i (\delta b_i)^2 < g\sum_{(i,j)=nn}(\delta\varphi_i - \delta\varphi_j)^2, \tag{21}$$

$$2E_L = k\sum_i (\delta b_i)^2 + g\sum_{(i,j)=nn}(\delta\varphi_i - \delta\varphi_j)^2 < 2W_L, \tag{22}$$

whenever $\Delta p > \Delta p_0$. Note that the inequality of Eq. (22) is not inconsistent with the inequality $W_L < 2E_L$ of Eq. (6). We also note that the inequality of Eq. (21) should suffice to ensure that the elastic network under consideration is in the universality class where *k* is irrelevant.

From Eqs. (7) and (8) we now get

$$E_L \propto \Delta p^{T-2\nu}, \quad W_L \propto \Delta p^{t}, \quad E_L < W_L \Rightarrow T \geqslant t+2\nu. \tag{23}$$

Together with Eq. (9), this leads to the result

$$T = t+2\nu. \tag{24}$$

It is generally believed that, even with networks of different topologies, the critical behaviors of randomly diluted versions of such networks are characterized by the same universal values of the critical exponents [23]. It follows that Eq. (24) should hold in all of those cases.

The network variational principles can also be applied to rigid/normal mixtures, so that we again obtain Eq. (6). However, in order to usefully apply this inequality to such a network that is near its rigidity threshold, we need to look at a different type of macroscopic boundary value problem: We consider a network of size $\xi \times \xi$ which is tied by normal bonds to a rigid or controlled boundary, as shown in Fig. 1(b). That boundary is displaced by an amount which is a linear function of position $\mathbf{u}_0 = \hat{\epsilon}_0 \cdot \mathbf{r}$, so that the macroscopic or average strain is $\hat{\epsilon}_0$. Because of its size, the system will typically consist of one large main rigid cluster which is connected to the boundary by a small number of normal (i.e., nonrigid) bonds. In fact, it was rigorously proven that the average total number of "singly disconnecting bonds (SDB)," between any two sites of the network separated by a distance ξ, (an SDB between two given sites of a rigid/normal or superconducting/normal conducting network is a normal bond that, when made rigid or superconducting, makes the connection between those sites rigid or superconducting; obviously, SDB in superelastic networks is the counterpart of SCB in diluted networks) is proportional to $1/|\Delta p|$ [28], i.e., it is large. Those SDB's will undergo length distortions which are comparable to the overall macroscopic displacement at the upper boundary of the system of Fig. 1(b), i.e., $\delta b \sim |\mathbf{u}_0| \sim \varepsilon_0 \xi$, where ε_0 represents the magnitude of the imposed macroscopic or average strain $\hat{\epsilon}_0$, and the angles adjacent to those SDB's will change by an amount of order $\delta\varphi_0 \equiv \varepsilon_0 \xi / b_0$. The total elastic potential energy will then be of order

$$E_L \sim C_e \xi^2 \varepsilon_0^2 = C_e b_0^2 (\delta\varphi_0)^2. \tag{25}$$

This differs from Eq. (7) in that the elementary unstressed bond length b_0 appears instead of the diverging percolation correlation length ξ. For the covering network of conductors $g_{ij} = m_{ij}$, with boundary condition $V_i = 0$ at the lower bar and $V_i = \delta\varphi_i \sim \delta\varphi_0$ at many other sites on the boundary, the total production rate of Joule heat is then of order

$$W_L \sim \sigma_e (\delta\varphi_0)^2, \tag{26}$$

which is identical with Eq. (8). Using these estimates in the inequality (6) we now get

$$C_e b_0^2 > \sigma_e \Rightarrow |\Delta p|^{-S} > |\Delta p|^{-s} \Rightarrow S \geqslant s. \tag{27}$$

Together with Eq. (3), this leads to the result

$$S = s. \tag{28}$$

As in the case of the diluted elastic network, rigid/normal networks of different topologies are expected to belong to the same universality class. Thus Eq. (28) should hold for all of them.

IV. SUMMARY AND DISCUSSION

The equalities $T = t+2\nu$, $S = s$, which were strongly indicated previously by numerical calculations of *T*, and by some numerical calculations of *S*, using 2D random networks with both bond stretching and angle bending forces, and by numerical calculations of $t = s$ in 2D random conducting networks, have now been proven rigorously, using variational principles for such networks. Although these equalities had been conjectured earlier, and were supported by some of the numerical simulations, a convincing proof was lacking until now. These results now join the set of other exact results for 2D percolating systems.

The bounds on *T* for diluted networks of arbitrary dimensionality *d*, namely, [see Eqs. (1) and (2)],

$$1 + d\nu \leqslant T \leqslant t + 2\nu,$$

are actually quite tight for $d = 2,3,4,5$, and the interval that they define actually shrinks to 0 when $d = 6$ [24]. The result $T = t+2\nu$, therefore, holds exactly both when $d = 2$ and when $d = 6$, and at least to a good approximation for values of *d* in between those two. It is natural to wonder whether $T = t+2\nu$, and also $S = s$, might perhaps be true for all values of *d*. Although the variational properties of the expressions for E_L and W_L of networks extend to arbitrary dimensionality, the detailed form of the expression for E_L becomes more complicated, since the orientation of a bond is determined by $d-1$ angles in the general case, and since the elastic potential energy must then depend also on angles between further than nearest neighbor bonds. That is why we are currently unable to generalize the derivation of the new lower bound for *T* [Eq. (23)] to other values of *d*. This remains a challenge for future studies. By contrast, it appears that the new lower bound for *S* [Eq. (27)] can be extended to arbitrary values of *d* [29].

We note that the arguments and proofs presented here do not rely on the fact that the fractal percolating cluster is produced by a random assignment of the network bonds to two different classes. Therefore they should also be applicable to other fractal systems, such as the one discussed in Ref. [20], which is based on the fractals known as Koch curves.

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[1] S. Feng and P.N. Sen, Phys. Rev. Lett. **52**, 216 (1984).
[2] D.J. Bergman and Y. Kantor, Phys. Rev. Lett. **53**, 511 (1984).
[3] D.J. Bergman, Phys. Rev. B **31**, 1696 (1985).
[4] J.G. Zabolitzky, D.J. Bergman, and D. Stauffer, J. Stat. Phys. **44**, 211 (1986).
[5] M.A. Lemieux, P. Breton, and A.-M.S. Tremblay, J. Phys. (France) Lett. **46**, L1 (1985).
[6] A.R. Day, R.R. Tremblay, and A.-M.S. Tremblay, Phys. Rev. Lett. **56**, 2501 (1986).
[7] L. Benguigui, Phys. Rev. Lett. **53**, 2028 (1984); J. Vareille, *ibid.* **57**, 1189 (1986); L. Benguigui, *ibid.* **57**, 1190 (1986).
[8] L. Benguigui, Phys. Rev. B **34**, R8176 (1986); J. Wu, E. Guyon, A. Palevski, S. Roux, and I. Rudnick, C.R. Acad. Sci., Ser. II: Mec., Phys., Chim., Sci. Terre Univers **305**, 323 (1987).
[9] L.C. Allen, B. Golding, and W.H. Haemmerle, Phys. Rev. B **37**, 3710 (1988).
[10] J.P. Straley, Phys. Rev. B **15**, 5733 (1977).
[11] D.J. Bergman and D. Stroud, Solid State Phys. **45**, 147 (1992).
[12] S. Feng, P.N. Sen, B.I. Halperin, and C.J. Lobb, Phys. Rev. B **30**, 5386 (1984).
[13] S. Roux, C. R. Acad. Sci., Ser. II: Mec., Phys., Chim., Sci. Terre Univers **301**, 367 (1985).
[14] D.J. Bergman, Phys. Rev. B **33**, 2013 (1986).
[15] S. Feng, Phys. Rev. B **32**, R510 (1985).
[16] S. Arbabi and M. Sahimi, Phys. Rev. Lett. **65**, 725 (1990).
[17] H.J. Herrmann, B. Derrida, and J. Vannimenus, Phys. Rev. B **30**, 4080 (1984).
[18] J.G. Zabolitzky, Phys. Rev. B **30**, 4077 (1984).
[19] C.J. Lobb and D.J. Frank, Phys. Rev. B **30**, 4090 (1984).
[20] L. Limat, Phys. Rev. B **40**, 9253 (1989).
[21] Y. Kantor and I. Webman, Phys. Rev. Lett. **52**, 1891 (1984).
[22] B. Nienhuis, J. Phys. A **15**, 199 (1982).
[23] A. Aharony and D. Stauffer, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1992).
[24] D.J. Bergman, in *Nonclassical Continuum Mechanics*, edited by R.J. Kuops and A.A. Lacey, London Mathematics Society Lecture Notes Series Vol. 122 (Cambridge University Press, Cambridge, England, 1987), p. 166.
[25] S. Roux, J. Phys. A **19**, L351 (1986).
[26] G.W. Milton, in *Physics and Chemistry in Porous Media*, edited by D.L. Johnson and P.N. Sen, AIP Conf. Proc. No. 107 (AIP, New York, 1984), P. 66–77.
[27] V.K.S. Shante and S. Kirkpatrick, Adv. Phys. **20**, 325 (1971).
[28] D. Wright, D.J. Bergman, and Y. Kantor, Phys. Rev. B **33**, 396 (1986).
[29] D.J. Bergman (unpublished).